Math 550 – David Dumas – Spring 2019

Problem Set 3

Due Monday, February 25 in class

Exercises: Work these out, but do not submit them.

(E1) Check that in both the topological and smooth categories, the operations of pullback and product over a fixed base for fiber bundles, as defined in lecture, yield fiber bundles.

(E2) Give example of a fiber bundle with typical fiber $\mathbb{R}^n$ for some n that is not isomorphic (as a fiber bundle) to any vector bundle.

(E3) Recall that S^3 is a principal S^1 bundle over S^2 by the Hopf fibration. For each integer n, the Lie group S^1 acts on itself by the rule $z \cdot w = z^n w$. Let S^1_n denote the space S^1 with this action of S^1. Show that the total space of the associated bundle $S^3(S^1_n)$ is diffeomorphic to the lens space $L(n,1)$.

(E4) Show that a principal bundle is trivial if and only if it has a section.

(E5) Show that a principal bundle map (i.e. morphism) over the identity $B \to B$ is necessarily a bundle automorphism.

Problems: Complete and submit <u>two</u> of these.

(P1) Let $\pi : P \to B$ be a principal G-bundle and F a space with an action of G. Prove that the two definitions of the F-bundle associated to P are equivalent, and that each gives a G-structure on a fiber bundle over B with typical fiber F. The definitions are:

- Global quotient definition: The group G has a left action on the product space $P \times F$ by $g \cdot (p, f) = (p \cdot g^{-1}, g \cdot f)$. Let $P(F) := (P \times F)/G$ denote the quotient space, and $\tilde{\pi} : P(F) \to B$ the map induced by $(p, f) \mapsto \pi(p)$. Then $(P(F), B, \tilde{\pi})$ is the associated fiber bundle.
- Local trivialization definition: Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of B with associated local trivializations of P given by $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$, and fiber transition maps $t_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. Define $P(F)$ as the quotient space of

$$\bigsqcup_{\alpha \in A} (U_\alpha \times F)$$

by the equivalence relation generated by

$$(x, f) \in (U_\beta \times F) \;\sim\; (x, t_{\alpha\beta}(x) \cdot f) \in (U_\alpha \times F)$$

where $x \in U_\alpha \cap U_\beta$. The projection map $\tilde{\pi} : P(F) \to B$ is induced by the maps $(x, f) \mapsto x$ on each $U_\alpha \times F$ component.

(P2) Let $\mathbb{RP}^n$ denote the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$, which is a homogeneous space of the Lie group $\mathrm{GL}_{n+1}\mathbb{R}$. There is a line bundle $\tau \to \mathbb{RP}^n$ which can be described in two ways; show that these two descriptions are equivalent.

- Let P denote the manifold $\mathrm{GL}_{n+1}\mathbb{R}$ considered as a principal H-bundle over $\mathbb{RP}^n$, where H is the stabilizer of a line $\ell \subset \mathbb{R}^{n+1}$. Then H acts on ℓ by linear maps, and so we have the associated vector bundle $\tau = P(\ell)$.
- Consider the subset $\tau \subset \mathbb{RP}^n \times \mathbb{R}^{n+1}$ defined by

$$\tau = \{(l,p) \,|\, p \in l\}.$$

Then τ is an embedded submanifold and the restriction of the projection $\pi_1 : \mathbb{RP}^n \times \mathbb{R}^{n+1} \to \mathbb{RP}^n$ gives τ the structure of a smooth line bundle.