Math 550 – David Dumas – Spring 2019

Problem Set 6

Due Monday, April 15 in class

Problems: Complete and submit three of these.

(P1) We have seen that for a Lie group G and closed subgroup H, we can view G as a principal H-bundle over the homogeneous space G/H. In this problem we use P to denote the manifold G when considered as a principal bundle over G/H.

Let $\mathfrak{g}/\mathfrak{h}$ denote the quotient vector space. The action of H on $\mathfrak{g}$ by the adjoint representation (of G, restricted to H) preserves $\mathfrak{h}$, and therefore induces a linear action of H on $\mathfrak{g}/\mathfrak{h}$. Using this, we can form the associated vector bundle $P(\mathfrak{g}/\mathfrak{h})$ over G/H.

Show that $P(\mathfrak{g}/\mathfrak{h})$ is isomorphic to the tangent bundle $T(G/H)$. That is, give a map and show it is a bundle isomorphism.

(P2) Let $G(k,n)$ denote the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$. For $W \in G(k,n)$, let $\Pi_W : \mathbb{R}^n \to W$ denote the orthogonal projection onto W with respect to the standard inner product on $\mathbb{R}^n$.

There is a vector sub-bundle τ of the trivial bundle $G(k,n) \times \mathbb{R}^n$ whose fiber τ_W over $W \in G(k,n)$ is simply $W \subset \mathbb{R}^n$.

Using the standard inner product on $\mathbb{R}^n$ we can define a connection on τ as follows: For $W \in G(k,n)$ let $\Pi_W : \mathbb{R}^n \to W$ denote the orthogonal projection. Let d denote the trivial connection on $G(k,n) \times \mathbb{R}^n$. For a section s of τ, which is in particular a section of $G(k,n) \times \mathbb{R}^n$, define

$$(\nabla s)(W) = \Pi_W \circ ds$$

that is, $\nabla(s)$ is the orthogonal projection onto τ of the covariant derivative of s with respect to the trivial connection.

(a) Show that ∇ is indeed a connection on τ.

(b) The manifold $G(k,n)$ can be identified with the homogeneous space $O(n)/(O(k) \times O(n-k))$, where $O(j)$ is the space of $j \times j$ orthogonal matrices. Given $U \subset G(k,n)$ and a section $\sigma : U \to O(n)$ of this quotient over U, describe how σ can be used to obtain:

(i) A local frame for τ over U, and

(ii) A formula for the connection matrix of ∇ relative to this frame in terms of the Maurer-Caran form $\omega_{O(n)}$.

(P3) A smooth map $f : M \to G(k,n)$ defines a sub-bundle E_f of the trivial vector bundle $M \times \mathbb{R}^n$ by

$$E_f = \{(p,v) \mid v \in f(p)\}.$$

The map $\Pi : M \times \mathbb{R}^n \to E_f$ given by $\Pi(p,v) = (p, \Pi_{f(p)}(v))$ is a vector bundle map over id_M.

Let ∇^0 denote the trivial connection on $M \times \mathbb{R}^n$.

(a) Given a section s of E_f define $\nabla^f s = \Pi \circ (\nabla^0 s)$. Show that ∇^f is a connection on E_f.

(b) How do E_f and ∇^f relate to the construction of the previous problem?