**Math 550 – David Dumas – Spring 2019**

# Problem Set 7

Due Monday, April 29 in class

**Exercises: Work these out, but do not submit them.**

(E1) Let $(V,\Omega)$ be a symplectic vector space (with $V$ finite-dimensional).
  (a) If $W \subset V$ is an isotropic subspace, show that there is a symplectic basis $e_i, f_i$ of $V$ in which $W = \mathrm{span}(e_1,\dots,e_k)$ for some $k$.
  (b) If $W \subset V$ is a coisotropic subspace, show that there is a symplectic basis $e_i, f_i$ of $V$ in which $W = \mathrm{span}(e_1,\dots,e_n,f_1,\dots,f_k)$ for some $k$.
  (c) Show that $W$ is isotropic if and only if $W^\perp$ is coisotropic.

(E2) Let $V$ be a vector space of dimension $2n$ and $\Omega$ an alternating bilinear form on $V$, i.e. $\Omega \in \bigwedge^2 V^*$. Show that $\Omega$ is symplectic if and only if $\Omega^{\wedge n} \in \bigwedge^{2n} V^* \simeq \mathbb{R}$ is nonzero.

(E3) Let $E$ be a finite-dimensional vector space. Let $V = E \oplus E^*$, and define $\Omega_E((x,\varphi),(y,\psi)) = \psi(x) - \varphi(y)$. Show that $\Omega_E$ is symplectic on $V$, and that $E \times \{0\}$ is Lagrangian.

(E4) Show that the previous construction is the universal description of Lagrangians: If $(V,\Omega)$ is symplectic and $L \subset V$ is Lagrangian, then $(V,\Omega)$ is symplectomorphic to $(L \oplus L^*, \Omega_L)$ by a symplectomorphism taking $L$ to $L \times \{0\}$.

(E5) Let $(M_i,\omega_i)$ be a symplectic manifold for $i = 1,2$. Let $\pi_i : M_1 \times M_2 \to M_i$ denote the projections onto the factors. Show that $\pi_1^*\omega_1 - \pi_2^*\omega_2$ is a symplectic form on $M_1 \times M_2$, and that a map $f : M_1 \to M_2$ is a symplectomorphism if and only if its graph is a Lagrangian submanifold of $M_1 \times M_2$.

**Problems: Complete and submit <u>three</u> of these.**

(P1) (Lee) Show that for a manifold $M$ of positive dimension, the tautological form on $T^*M$ is not the pullback of any differential form on $M$.

(P2) It follows from our discussion of Hamiltonian vector fields that symplectic manifolds have many infinitesimal symmetries; for example, if $\alpha \in \Omega^2(M)$ is a symplectic form, then we have:

$(*)$ For all $(x,v) \in TM$, there exists $V \in \mathrm{Vect}(M)$ with $V_x = v$ and $\mathscr{L}_V \alpha = 0$.

  (a) Work out the details to show that $(*)$ holds for symplectic forms $\alpha$.

How specific is $(*)$ to symplectic forms? That is, one can ask whether $(*)$ holds for any differential form $\alpha \in \Omega^k(M)$. Determine whether it holds for the following classes:

  (b) Closed 2-forms
  (c) Nondegenerate 2-forms
  (d) Nowhere zero 2-forms
  (e) Closed 1-forms
  (f) Nowhere zero 1-forms
  (g) Nowhere zero closed 1-forms

(P3) (a) Let $(M,\omega)$ be a compact symplectic manifold. Show that $H^{2k}(M) \neq 0$ for all $k \leq \frac{1}{2}\dim(M)$.

(b) Give an example of a compact manifold $M$ of even dimension with $H^{2k}(M) \neq 0$ for all $k \leq \frac{1}{2}\dim(M)$ but which does not have a symplectic form.