Ext exact sequence

Fix an abelian group G (the "coefficient group"). We've seen that a short exact sequence of abelian groups

$$0 \to A \to B \to C \to 0$$

induces a dual exact sequence

$$\mathrm{Hom}(A,G) \leftarrow \mathrm{Hom}(B,G) \leftarrow \mathrm{Hom}(C,G) \leftarrow 0$$

but in general one cannot add "$0 \leftarrow$" at the left of this and retain exactness. That is, the map to $\mathrm{Hom}(A,G)$ need not be surjective. We then introduced free resolutions and the Ext functor. However we did not complete the original discussion by describing how to continue the dual exact sequence above.

In fact, what one obtains is an exact sequence:

$$\begin{tikzcd}
& \mathrm{Hom}(A,G) \arrow[drr] & \mathrm{Hom}(B,G) \arrow[l] & \mathrm{Hom}(C,G) \arrow[l] & 0 \arrow[l] \\
0 & \mathrm{Ext}^1(A,G) \arrow[l] & \mathrm{Ext}^1(B,G) \arrow[l] & \mathrm{Ext}^1(C,G) \arrow[l] &
\end{tikzcd}$$

Recalling that $\mathrm{Hom}(A,G) = \mathrm{Ext}^0(A,G)$, this is an exact sequence with three Ext^0 terms and three Ext^1 terms. It is thus reminiscent of the H^0 and H^1 part of the long exact sequence in cohomology that arises from a short exact sequence of cochain complexes. Indeed, that is how it is constructed: The exact sequence $0 \to A \to B \to C \to 0$ can be turned into an exact sequence of free resolutions of those groups (that is, a short exact sequence of c free chain complexes). After taking duals, one has a short exact sequence of cochain complexes. The associated long exact sequence of cohomology is the Ext exact sequence above.

The final surjection $0 \leftarrow \mathrm{Ext}^1(A,G)$ in the diagram above arises because abelian groups have 2-step free resolutions, so $\mathrm{Ext}^i(-,G)$ is zero for $i > 1$. In a more general setting (e.g. modules over a commutative ring and Ext computed using *projective* resolutions) the sequence would continue with terms $\mathrm{Ext}^i(-,G)$ for all $i \geq 0$.