Math 547: Algebraic Topology I – David Dumas – Fall 2023

Homework 12

Due Wednesday November 29 at 11:59pm

Instructions: *Same as in Homework 2.*

Problems:

- 2.2.28
- 2.2.29 (Note: a 3-manifold is a second countable space locally homeomorphic to $\mathbb{R}^3$. While the problem uses this term, you don't really need to know anything about 3-manifolds to solve the problem.)

(P1) Let $V_1, \dots, V_n, W_1, \dots, W_n$ be finite dimensional vector spaces over $\mathbb{R}$ and $T_i : V_i \to W_i$ a collection of $\mathbb{R}$-linear maps. Suppose that for each i there is a choice of bases for V_i and W_i so that the matrix of T_i has integer entries.

(a) (4 points) Construct compact, connected spaces X, Y and a continuous map $f : X \to Y$ such that for $1 \leqslant i \leqslant n$ we have $\tilde{H}_i(X;\mathbb{R}) \simeq V_i$ and $\tilde{H}_i(Y,\mathbb{R}) \simeq W_i$ and a commutative diagram

$$\begin{array}{ccc} \tilde{H}_i(X) & \xrightarrow{f_*} & \tilde{H}_i(Y) \\ \sim\downarrow & & \downarrow\sim \\ V_i & \xrightarrow[T_i]{} & W_i \end{array}$$

That is, show that you can realize any such collection of finite-dimensional vector spaces and linear maps using homology with $\mathbb{R}$ coefficients.

(b) ($\frac{1}{2}$ point) Show furthermore that there are infinitely many homeomorphism classes of spaces that can be used as X and Y in part (a).

(c) ($\frac{1}{2}$ point) For any natural number k show there exist compact connected spaces X and Y such that there are at least k different homotopy classes of continuous maps $f : X \to Y$ that can be used in part (a).

(P2) Let Σ_g denote the orientable surface of genus g. Show that there are infinitely many homotopy classes of maps $\Sigma_g \to S^2$.