Math 547: Algebraic Topology I – David Dumas – Fall 2023

Homework 3

Due Monday September 18 at 11:59pm

Instructions: *Same as in Homework 2.*

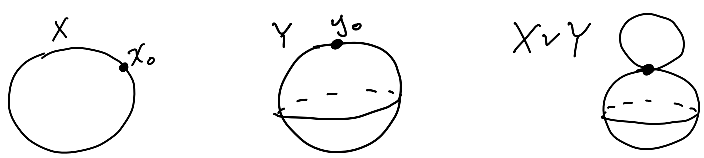
Note that this assignment corresponds to course material up to and including Monday September 11. That means in particular you cannot use Van Kampen's theorem in your solutions.

Problems: ($*$ means expected to be more challenging)

– 1.1.16 parts (a),(b),(c),(d).

– 1.1.17

Note these problems use a notation we haven't yet discussed in lecture: If X and Y are spaces with basepoints $x_0 \in X$ and $y_0 \in Y$, then $X \vee Y$ is the quotient of $X \sqcup Y$ by the equivalence relation generated by $x_0 \sim y_0$. In other words, you glue the basepoint in X to the basepoint of Y but leave everything else alone.



(P1) A *topological group* is a pair consisting of a group G and a topology on the underlying set of G such that the maps $\mu : G \times G \to G$, $\mu(g,h) = gh$, and $\iota : G \to G$, $\iota(g) = g^{-1}$ are continuous with respect to the given topology.

Let G be a topological group with identity element $e \in G$. Show that the group $\pi_1(G,e)$ is abelian.

Hint: Because G is a group there are additional ways to multiply two paths γ and η, different from concatenation. You could consider the "pointwise product" paths $s \mapsto \mu(\gamma(s),\eta(s))$ and $s \mapsto \mu(\eta(s),\gamma(s))$.

(P2) $*$ **Dendritic wedge sum of simply connected locally contractible pieces.**

First, a little background. In class we discussed approaches to proving $\pi_1(S^n) = 0$ for $n > 1$ based on smoothing or by reducing it to a topological lemma we didn't prove (i.e. every loop is homotopic to a non-surjective one).

Hatcher's argument for $\pi_1(S^n) = 0$ if $n > 1$ is different, and is based on a nice lemma. This problem is designed to make you read and think about the proof of that lemma. Here's the lemma itself:

Lemma 1 (Hatcher's Lemma 1.15)**.** *If a space X is a union of a collection of path-connected open sets A_α each containing the basepoint $x_0 \in X$, and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then every loop in X at x_0 is homotopic to a product of loops each of which is contained in a single A_α.*

This lemma is helpful because it lets you "localize" a loop, homotoping it to a finite concatenation of loops that stay in sets you have some control over (the A_α).

Look over the proof[1], and pay attention to the way the condition that every set contains the basepoint is used. In this problem, you may want to use similar methods but without that condition in place. (Hatcher's problem 1.1.19 covers some similar ground.)

A space. First, if E is a set we let $\mathrm{NRW}(E)$ denote the set of finite tuples of elements of E in which no element of E appears twice in a row. For convenience we'll call elements of $\mathrm{NRW}(E)$ *words*, rather than tuples, and write them without parentheses or commas. (NRW stands for non-repeating words.) We allow the empty word here, which we denote by $-$. So if $E = \{A, B, C\}$ then $\mathrm{NRW}(E)$ contains elements such as $-$, A, B, C, AB, BA, CB, CBC, $ABABACACABABACBCB$, etc.

Next, let Y be a Hausdorff topological space. Let E be a nonempty finite subset of Y. Thus $\mathrm{NRW}(E)$ is a countably infinite set. Let $\mathbb{Y}$ denote the topological space that is an infinite disjoint union of copies of Y, one for each element of $NRW(E)$:

$$\mathbb{Y} = \bigsqcup_{s \in \mathrm{NRW}(E)} Y_s$$

(This is another way of saying $\mathbb{Y} = Y \times \mathrm{NRW}(E)$ with $\mathrm{NRW}(E)$ having the discrete topology, and that we use the notation Y_s instead of $Y \times \{s\}$.)

Note that since $E \subset Y$, for each $x \in E$ and $s \in \mathrm{NRW}(E)$ there is a corresponding point in Y_s. We denote this by $x_s \in Y_s$.

Now we introduce an equivalence relation $\sim$ on $\mathbb{Y}$. Suppose $s \in \mathrm{NRW}(E)$ and $x \in E$. Let sx denote the word that is obtained from s by adding x at the end (which is possible as long as s doesn't end with x). Now, any time $s, s' \in \mathrm{NRW}(E)$ are related by $s' = sx$ we declare that $x_s \sim x_{s'}$. Let $\sim$ be the equivalence relation generated by these conditions.

Now let $X = \mathbb{Y}/\sim$ with the quotient topology. In words, X is obtained by $\mathbb{Y}$ by gluing some pairs of copies of Y together; specifically, whenever Y_s and $Y_{s'}$ correspond to words that differ only by appending a letter at the end, you glue them together at the point corresponding to that letter.

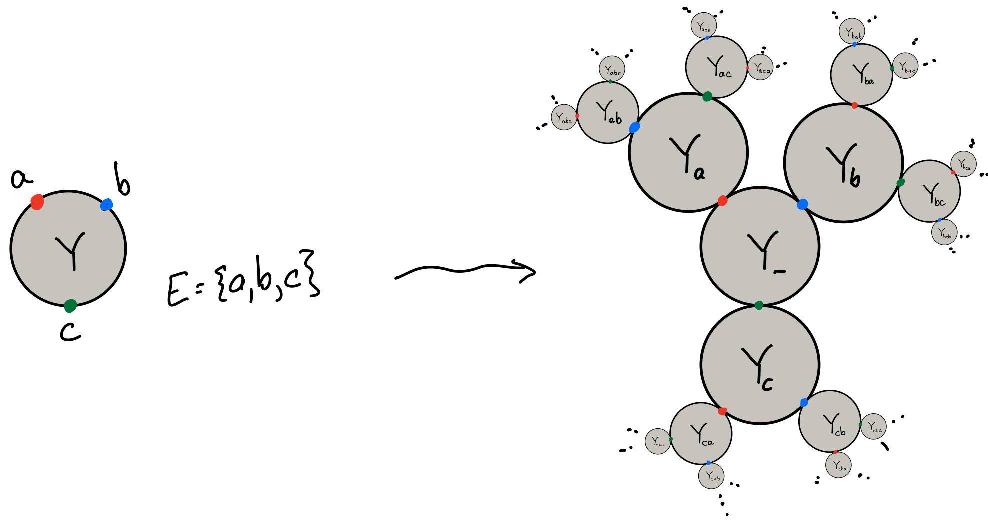
The problem. Suppose Y is a space that is simply connected and which locally deformation retracts onto its points, meaning for any $y \in Y$ and for any neighborhood U of y there exists an open set V with $y \in V \subset U$ such that V deformation retracts onto $\{y\}$. Show that for any finite $E \subset Y$ the space X constructed as above has $\pi_1(X) = 0$.

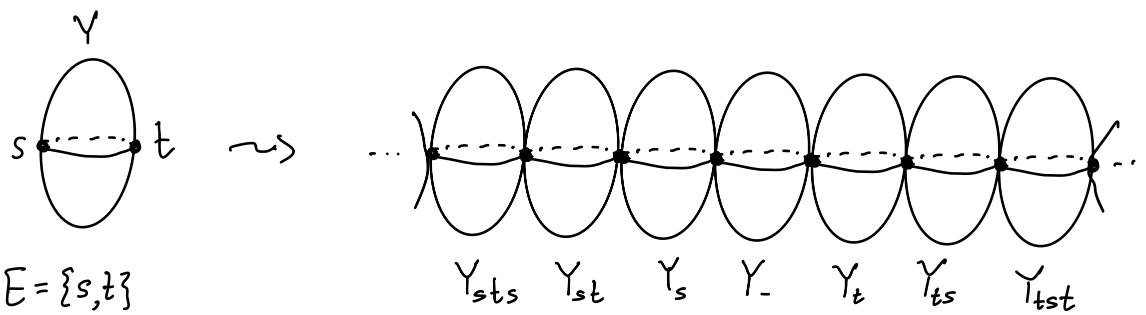
Some pictures. On the next page are some pictures of this construction for various Y and $E \subset Y$.

Revision history:

- 2023-09-14 Replace locally contractible with a stronger condition that is required for the method I had in mind.

[1] And if you find the lemma and proof missing from your textbook, note that versions of the text from before 2015 are slightly different; the latest version has it and is freely available online.


a
b
c
Y
E = {a,b,c}
Y_a
Y_b
Y_-
Y_c
Y_ab
Y_ac
Y_ba
Y_bc
Y_ca
Y_cb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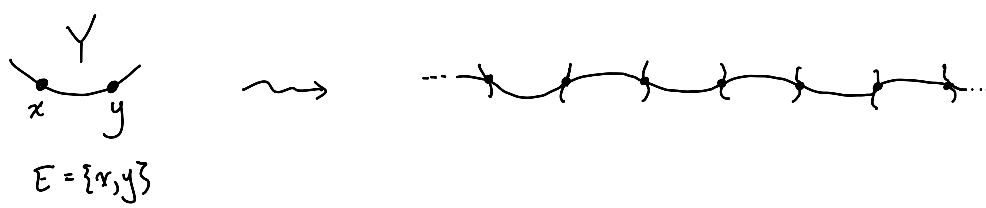
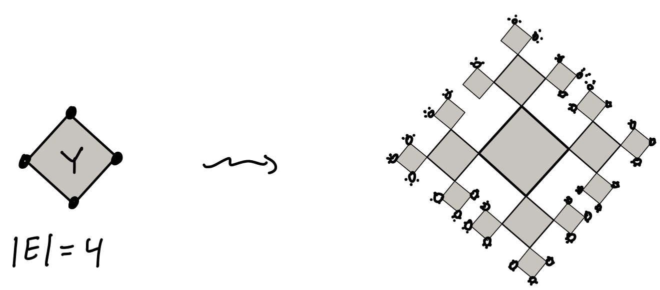
t
E = {s,t}
Y_sts
Y_st
Y_s
Y_-
Y_t
Y_ts
Y_tst


Y
x
y
E = {x,y}


Y
|E| = 4