Math 547: Algebraic Topology I – David Dumas – Fall 2023

Homework 5

Due Monday October 2 at 11:59pm

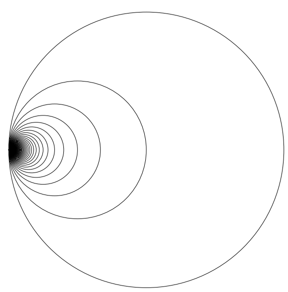
Instructions: *Same as in Homework 2.*

Problems:

(P1) Let Y be a connected space and $p : Y \to X$ a 2-sheeted covering. Thus for each $y \in Y$ the set $p^{-1}(p(y))$ consists of two points, one of which is y. Denote the other one by $f(y)$, so that $f : Y \to Y$ is a map of sets that is an involution ($f \circ f = \mathrm{Id}_Y$). Show that the map f thus defined is continuous and defines an automorphism of the covering. Also show this automorphism generates $\mathrm{Aut}(Y/X)$ which is therefore isomorphic to $\mathbb{Z}/2$.

(P2) Let $p : \tilde{X} \to X$ be a covering map with $\tilde{X}$ path-connected. Let $x_0 \in X$, and let $y, z \in p^{-1}(x_0)$. Show that the subgroups $p_*(\pi_1(\tilde{X}, y))$ and $p_*(\pi_1(\tilde{X}, z))$ of $\pi_1(X, x_0)$ are conjugate.
(Based on an exercise in Fulton's book.)

For these next problems, let C_n denote the circle in $\mathbb{R}^2$ of radius $\frac{1}{n}$ in $\mathbb{R}^2$ centered at $(\frac{1}{n}, 0)$. Let $E = \cup_{n=1}^{\infty} C_n$, with the subspace topology. The space E is often called the *earring* or *infinite earring*. Hatcher calls it the *The Shrinking Wedge of Circles*.You can find a picture of it below, and it is also discussed in Example 1.25 in Hatcher's book (though that's in the section on Van Kampen's theorem, which we haven't covered yet).



Also, in these problems we will say that a covering space $p : \tilde{X} \to X$ *unwraps* a loop $\gamma : I \to X$ if some lift of γ to a path in $\tilde{X}$ has distinct endpoints.

(P3) Show that the earring E has no simply-connected covering space. Conclude that there is no covering space of E that unwraps every loop in E.

(P4) Let γ_n be a loop in E based at $(0,0)$ that traverses the circle C_n once. Show that for each n there is a finite-sheeted covering space that unwraps E.

Revision history:

- 2023-09-29 Add that Y is connected in P1