Math 547: Algebraic Topology I – David Dumas – Fall 2023

Homework 8

Due Monday October 30 at 11:59pm[1]

Instructions: *Same as in Homework 2.*

Problems:

– 2.1.5

– 2.1.8

(P1) Let $\mathscr{U} = \{U_i\}_{i \in A}$ be an open cover of a space X, where elements of the cover are labeled by elements of a set A. Suppose a total order $<$ has been specified on A (for example we might have $A = \mathbb{Z}$ with the usual ordering), making this an *ordered open cover*. There is a Δ-complex we can then associate to $\mathscr{U}$, called the *nerve of* $\mathscr{U}$, defined as follows. First, let $A^{[n]}$ denote the collection of n-tuples $(i_1, \dots, i_n) \in A^n$ that are increasing, i.e. $i_1 < i_2 < \cdots < i_n$.For $I = (i_1, \dots, i_n) \in A^{[n]}$, let U_I denote the intersection

$$U_I = \bigcap_{k=1}^{n} U_{i_k}$$

which is a (possibly empty) open subset of X. Let $K^n = \{I \in A^{[n]} : U_I \text{ is not empty}\}$, so as n varies the sets K^n tell you which finite intersections of $\mathscr{U}$ are nonempty.

The nerve $N(\mathscr{U})$ is a Δ-complex in which there is one n-cell for each element of K^{n+1}. If $(i_0, \dots, i_n)$ is an element of K^{n+1} then we use these elements of A as labels for the vertices of the corresponding n-cell, denoting it $[i_0, \dots, i_n]$. The face $[i_0, \dots, \widehat{i_k}, \dots, i_n]$ is glued to the $(n-1)$-simplex corresponding to $(i_0, \dots, \widehat{i_k}, \dots, i_n) \in K^{n-1}$. Note that $(i_0, \dots, \widehat{i_k}, \dots, i_n)$ lies in K^{n-1} because $U_{(i_0, \dots, \widehat{i_k}, \dots, i_n)}$ contains $U_{(i_0, \dots, i_n)}$ and is thus nonempty.

Thus, in $N(\mathscr{U})$ there is a point for each open set in the cover, an edge for each pair of open sets that intersect nontrivially, a triangle for each triple that intersect nontrivially, etc.

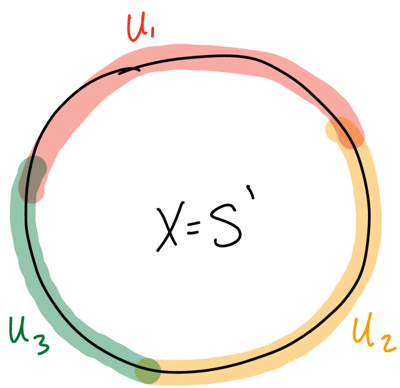
The simplicial homology $H_n^\Delta(N(\mathscr{U}))$ of the nerve is called the *Čech homology* of X relative to the open cover $\mathscr{U}$, denoted $\check{H}_n(X, \mathscr{U})$. (There's also a way to take a limit over finer and finer open covers and get an object $\check{H}_n(X)$ that doesn't depend on a particular open cover, but we won't use that here.)

On the next page are some pictures of finite open covers of spaces with labels taken from $\mathbb{Z}$ (thus giving an order). In each case, compute the associated Čech homology groups in all degrees.

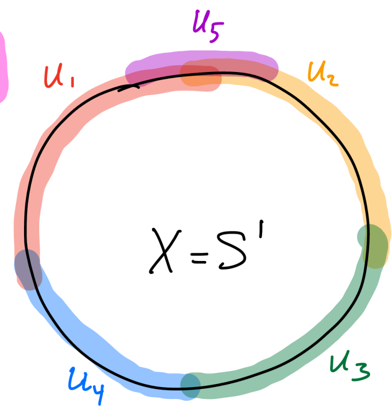
Figure on next page →

[1]And that's really the intended deadline this time!

(a)

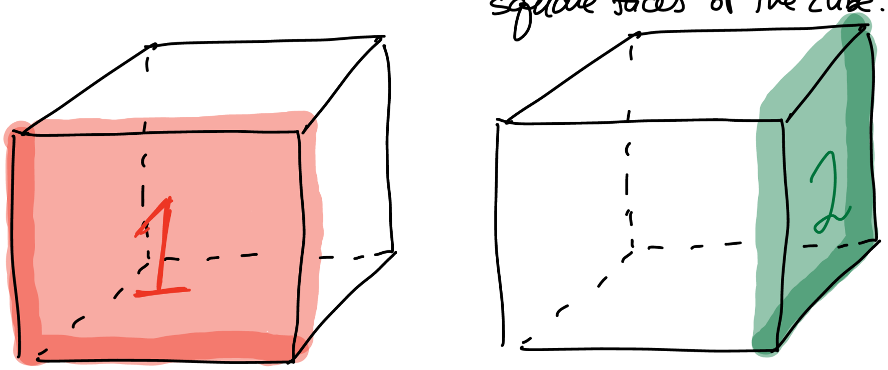


(b)



(c) $X = S^2 \cong$ surface of unit cube in $\mathbb{R}^3$

$\mathcal{U} = \{U_1, \dots, U_6\}$ $\quad U_i$ = open nbd of one of the square faces of the cube.



& similarly for the other faces.