Math 547: Algebraic Topology I – David Dumas – Fall 2023

Midterm Exam

Due Monday October 16 at 11:59pm

No collaboration ⋆ Consult only course notes and texts ⋆ See syllabus for full rules

(P1) For a subset $K \subset S^1$, define $C(K) \subset \mathbb{R}^2$ by $C(K) = \{cv : c \geq 0, v \in K\}$. Thus $C(K)$ is the union of all the rays from 0 in the directions of points of K.

(a) If K is finite, show that there exists a retraction $\mathbb{R}^2 \to C(K)$.

(b) Give an example of a closed set $K \subset S^1$ such that there is no retraction $\mathbb{R}^2 \to C(K)$ (and prove your example has this property).

(P2) Let ℓ_1 and ℓ_2 be distinct lines in $\mathbb{R}^3$. (Note that line does *not* mean linear subspace here; these lines are not required to contain the origin.) Show that $\mathbb{R}^3 \setminus (\ell_1 \cup \ell_2)$ is homotopy equivalent to either $S^1 \vee S^1$ or $S^1 \vee S^1 \vee S^1$, depending on whether or not the lines are disjoint.

(P3) Let X denote the set of 2-element subsets of S^1. We can put a topology on X as follows: Let Y be the set of ordered pairs of distinct points on S^1, $Y = \{(a,b) \in S^1 \times S^1 : a \neq b\}$. Give Y the subspace topology as a subset of $S^1 \times S^1$. Finally, identify X with the quotient of Y by the equivalence relation generated by $(a,b) \sim (b,a)$, and give it the quotient topology.

(a) Show that the quotient map $Y \to X$ is a covering map.

(b) Let $A \subset X$ denote the set of 2-element subsets of S^1 that consist of diametrically opposite points (i.e. the line in $\mathbb{R}^2$ they determine passes through the origin). Show that X deformation retracts onto A.

(c) Compute $\pi_1(X)$ and describe the subgroup that corresponds to the cover $Y \to X$ in the classification of connected covering spaces.

(P4) Give explicit examples (with proof) of the following phenomena:

(a) Covering maps $p : Y \to X$ and $q : Z \to X$ such that Y and Z are homeomorphic, but there is no isomorphism of covering spaces from $Y \xrightarrow{p} X$ to $Z \xrightarrow{q} X$.

(b) G-covers $p : Y \to X$ and $q : Z \to X$ (for the same group G) that are isomorphic as covering spaces but not isomorphic as G-covers.