Opers on Riemann surfaces: Exercises

Emily Dumas

Key to labels: "⋈" = Explores connections to another topic, and difficulty will depend on familiarity therewith; "conj" = The question involves conjecture or speculation; "hard", "easy" = Difficulty estimates; "??" = I have not worked this out (but believe there is nothing new or conjectural here); "…" = this exercise continues a theme developed in the previous exercise, and may use notation introduced there.

(1) [easy] Verify the following relation between developing maps of projective structures and the maps between graded components induced by the connection, as mentioned in lecture: Let (f,ρ) be a $\mathbb{CP}^1$ structure on X. Let ∇ be the flat connection on $V := \tilde{X} \times_\rho \mathbb{C}^2$. Let $V_1 \subset V$ be the line bundle represented by f. Then the map

$$\bar{\nabla} : V_1 \to K \otimes (V/V_1)$$

can equivalently be considered as a map $TX = K^* \to \mathrm{Hom}(V_1, V/V_1)$. Using the natural isomorphism $T_\ell \mathbb{CP}^1 \simeq \mathrm{Hom}(\ell, \mathbb{C}^2/\ell)$ and that V_1 represents f, we an interpret $\mathrm{Hom}(V_1, V/V_1)$ as the pullback of $T\mathbb{CP}^1$ by f. In this way $\bar{\nabla}$ becomes a bundle map $TX \to f^*(T\mathbb{CP}^1)$. The map thus obtained is df.

(2) [⋈] Show that the jet bundle of a vector bundle V over X (as defined using germs of local sections) can also be described as

$$J^n(V) = (\pi_1)_* \left((\mathscr{O}_{X\times X}/\mathscr{I}_\Delta^{n+1}) \otimes \pi_2^* V \right)$$

where $\Delta \subset X \times X$ is the diagonal and $\mathscr{I}_\Delta \subset \mathscr{O}_{X\times X}$ is the sheaf of functions that vanish on Δ.

(3) What can be said about opers for genus $g \leqslant 1$? For $\mathrm{SL}(2,\mathbb{C})$ show that the trivial connection on $\mathscr{O} \oplus \mathscr{O}$ is an oper connection for the filtration given by the exact sequence

$$0 \to \mathscr{O}(-1) \to \mathscr{O} \oplus \mathscr{O} \to \mathscr{O}(1) \to 0.$$

(4) (I. Biswas) Suppose a projective structure on X is given, that z is a local projective coordinate, and that θ is a spin structure ($\theta^2 \simeq K$) with a chosen local frame $dz^{1/2}$ satisfying $dz^{1/2} \otimes dz^{1/2} = dz$. Show that there is a well-defined isomorphism

$$J^a(\theta^{-b}) \to J^a(\theta^{-a}) \otimes \theta^{a-b}$$

defined locally by

$$j^a_p\left((z-p)^k \left(\frac{\partial}{\partial z}\right)^{b/2} \right) = \frac{(a)_k}{(b)_k}\, j^a_p\left((z-p)^k \left(\frac{\partial}{\partial z}\right)^{a/2} \right) \otimes (dz)_p^{(b-a)/2},$$

where $(n)_k = n(n-1)(n-2)\cdots(n-k+1)$. That is, check that the same map is obtained if one uses a different projective coordinate $w = \frac{az+b}{cz+d}$. On the other hand, show that the local maps obtained in this way from arbitrary holomorphic coordinates (i.e. not projectively related) do not give a well-defined map.

(5) [⋈ conj] The Hitchin fibration is a proper surjective map $\pi : \mathrm{Hom}(\pi_1 X, \mathrm{SL}_n \mathbb{C}) \to \mathscr{B}_n(X)$. Is the set of holonomy representations of $\mathrm{SL}_n\mathbb{C}$-opers a section of this fibration?

(6) … Composing the parameterization of opers by $\mathscr{B}_n(X)$ with the Hitchin fibration gives a map $\mathscr{B}_n(X) \to \mathscr{B}_n(X)$. Is this map close to the identity, in some sense? (For $n = 2$ it not the identity map, but it is known that this map is $\phi \mapsto -\phi + o(\|\phi\|)$.)

(7) [easy] Verify the following description of the Schwarzian derivative due to Thurston: Given a holomorphic function f on a domain $U \subset \mathbb{C}$ with nowhere-vanishing derivative, for each $p \in U$ there exists a unique linear fractional transformation $m_{f,p} \in \mathrm{PSL}_2\mathbb{C}$ such that $j^2_p(m_{f,p}) = j^2_p(f)$. This gives a holomorphic map $U \to \mathrm{PSL}_2\mathbb{C}$, $p \mapsto m_{f,p}$. The pullback of the Maurer-Cartan 1-form of $\omega_{\mathrm{PSL}_2\mathbb{C}}$ by this map is a holomorphic 1-form $s_f \in \Omega^1(U, \mathfrak{sl}_2\mathbb{C})$. This form can be written as

$$s_f^*(\omega_{\mathrm{PSL}_2\mathbb{C}}) = \frac{1}{2} g \begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix} dz$$

where $S(f) = g\,dz^2$. That is, the pullback is a pointwise scalar multiple of a 1-form that does not depend on f, and that scalar is essentially the Schwarzian of f.

Date: June 20, 2024

(8) … Interpreting $\mathfrak{sl}_2\mathbb{C}$ as the algebra of holomorphic vector fields on $\mathbb{CP}^1$, the scalar multiples of $\begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix}$ correspond to the vector fields that vanish only at p (i.e. have a double zero there). Denote this line by P_z, so that the union of these gives a line bundle P over U embedded in $U \times \mathfrak{sl}_2\mathbb{C}$. Show that there is a natural isomorphism between T^*U and P which maps dz to $\frac{1}{2}\begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix}$. Thus, using this isomorphism the 1-form $s_f j(\omega_{\mathrm{PSL}_2\mathbb{C}})$ becomes a quadratic differential, which is $S(f)$.

(9) Let $f : \tilde{X} \to \mathbb{CP}^n$ be a holomorphic curve corresponding to an oper on X. Recall that successive derivatives of f then give a full flag at each point. Let $f^*(p)$ denote the codimension-1 component of this flag, i.e. the osculating hyperplane of f at p. Thus $f^* : \tilde{X} \to (\mathbb{CP}^n)^*$ is a curve in the dual projective space. Show that f^* is also associated to an oper on X. What is the relation between the tuple of holomorphic differentials describing f and those describing f^*?

(10) Let (V_i, ∇) be an oper on X. The dual bundle V^* has a filtration in which $V_i^* = V_{n-i}^{\perp}$. Show that (V_i^*, ∇^*) is also an oper on X? If so, what is the relation between the tuple of holomorphic differentials describing (V_i, ∇) and those describing (V_i^*, ∇^*)?

(11) [??] Let $D : J^n(L) \to K^n \otimes L$ denote a differential operator associated to an oper on X, i.e. the symbol is the identity and $\det J^n(L)$ is trivial. What is the duality/adjoint operation to obtain a differential operator $D^* : J^n(L) \to K^n \otimes L$ associated to the dual holomorphic curve?

(12) [$\bowtie$ conj] Are opers close to Veronese ones in a differential sense also close in a geometric sense? Here is a precise version. Define a norm on $\mathscr{B}_n(X)$ as follows: Let h denote the Kähler form of the hyperbolic metric of X. For $\phi_p \in H^0(K_X^p)$ let $\|\phi_p\|_\infty = \sup_X h^{-p/2}|\phi_p|$. For $\vec{\phi} = (\phi_2, \ldots, \phi_n) \in \mathscr{B}_n(X)$ let $\|\vec{\phi}\|_\infty = \max_p \|\phi_p\|_\infty$. Does there exist a universal constant b so that if $\|\vec{\phi}\|_\infty < b_n$ then the holonomy of the associated oper on X is a quasi-isometric embedding into $\mathrm{SL}_n\mathbb{C}$? Nehari's theorem and the Ahlfors-Weill extension say that one can take $b_2 = \frac{1}{2}$, for if $\|\phi_2\|_\infty < \frac{1}{2}$ then there is a quasiconformal conjugacy between the holonomy action on $\mathbb{CP}^1$ and that of the Fuchsian group uniforming X.

(13) Take the Veronese $\mathrm{SL}_n\mathbb{C}$-oper arising from the symmetric product of a projective structure (f, ρ). This gives an isomorphism of V with $\tilde{X} \times_\rho \mathbb{C}[x,y]_{n-1}$ where $\mathbb{C}[x,y]_{n-1}$ is the n-dimensional irreducible representation where $\mathrm{SL}_2\mathbb{C}$ acts on homogeneous polynomials of degree $n-1$. Let z be a projective coordinate on X. Compute the matrix representation of a p-differential ϕ_p (considered as a 1-form with values in $\mathrm{End}(V)$) with respect to the local frame $(x^{n-1}, x^{n-2}y, \ldots, xy^{n-2}, y^{n-1})$ in a few cases. (Suggested cases: all $n, p \leqslant 3$ and $n = 4, p = 3$.

(14) … Also find a local expression for the differential operator corresponding to $\nabla + \phi_p$ in these cases. (Compare to Equation 4.5 in Wentworth, "Higgs bundles and local systems on Riemann surfaces".)

(15) (R. Wentworth) Show that the choice of a spin structure θ (i.e. bundle with $\theta^2 \simeq K$) is precisely equivalent to lifting a Fuchsian group uniformizing X from $\mathrm{PSL}_2\mathbb{C}$ to $\mathrm{SL}_2\mathbb{C}$, in the following way: A lift $\rho : \pi_1 X \to \mathrm{SL}_2\mathbb{C}$ corresponds to the spin structure θ so that $(\tilde{X} \times_\rho \mathbb{C}^2)^* \otimes \theta^*$ has a holomorphic section that is not identically zero.

(16) [hard] Let V_i be the filtered holomorphic vector bundle underlying all $\mathrm{SL}_n\mathbb{C}$ opers. Let $W = \mathrm{gr}(V)$ be the associated graded bundle, i.e.

$$W = \oplus_i V_i / V_{i-1}.$$

Note that W inherits a holomorphic structure. Give an explicit formula for a $(0,1)$-form δ with values in $\mathrm{End}(W)$ so that $\bar{\partial}_W + \delta$ is another holomorphic structure on W such that $(W, \bar{\partial}_W + \delta) \simeq V$. (Hint: Use the hyperbolic metric on X in constructing δ.)

(17) [hard] … Now compute the endomorphism-valued 1-form corresponding to ϕ_p in the $(W, \bar{\partial}_W + \delta)$ model of V.

(18) [easy] (G. Anderson) For a compact Riemann surface X of genus g there is the canonical embedding $X \to \mathbb{CP}^{g-1}$. Does this ever arise from an oper structure? If so, what differentials describe its difference from the Veronese oper associated with uniformization of X?